Small-scale convection in the upper mantle beneath the Chinese Tian Shan Mountains

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Abstract

Small-scale convection of the upper mantle beneath the Chinese Tian Shan (Tien Shan) is investigated in terms of numerical modeling. The finite element method combined with the marker-in-cell technique is used to describe the flow of the heterogeneous upper mantle. The density model is derived from the P-wave velocity structure of the crust and upper mantle along the Kuche–Kuitun profile across the Chinese Tian Shan, which is obtained using the seismic travel time tomography technique. Our computational results reveal the southward-counterclockwise and northward-clockwise upper mantle convective cells underneath the Junggar-north Tian Shan and Tarim-south Tian Shan, respectively. Our results also show the convective scale reaches to $\sim 500 \text{ km}$ and the convective speed at the top of the upper mantle should not be less than 20 mm/year for a normal viscosity model. The northward extrusion of the Tarim block plays a key role in the Tian Shan mountain building since the Cenozoic period, but it nearly does not influence the upper mantle convection. The present-day tectonic deformation in the Chinese Tian Shan is related to the small-scale convection of the upper mantle.

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1. Introduction

The Tian Shan range is the largest and present-day most active intracontinental orogenic belt in the world (Fig. 1a). The Tian Shan dynamics has a great value for understanding the basic questions of continental dynamics, especially, the intracontinental deformation and its mechanism.

The recent GPS measurements show that the shortening rate across the western Tian Shan ($76^\circ\text{E}$) is $\sim 20 \text{ mm/year}$ and that across the eastern Tian Shan ($87^\circ\text{E}$) is only $\sim 3.5 \text{ mm/year}$ (Abrakhmatov et al., 1996; Wang et al., 2000). This attests to the fact that the crustal shortening rate of the Tian Shan range decreases eastward, although its mechanism is still poorly understood. Especially, it is also unclear how the deformation measured on the surface is related to the deformation within the crust and upper mantle (Vinnik et al., 2002). Usually, the rejuvenation of the Tian Shan since Miocene is attributed to the collision between the India and Eurasia plate and the effects conveyed though the Tarim block (Molnar and Tapponnier, 1975; England and Houseman, 2007).

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Fig. 1. The seismic station map across the Chinese Tian Shan (a) and the P-wave velocity perturbation in the crust and upper mantle along the Kuche–Kuitun profile (b). The topography of the Tian Shan and adjacent area is shown in the top-left of (a) and the frame with black solid line indicates the research area in this study. The black solid line in (b) represents the Moho discontinuity, which is taken from that by Li et al. (2007). The IASP91 P-wave velocity model is shown in the right side of (b).
sounding and 3D seismic tomography (Xu et al., 2000; Zhao et al., 2003; Xiao et al., 2004). However, these results just offered the structure down to the depth of 300 km or presented the structure of the upper mantle with horizontal resolution larger than 165 km, which is not satisfied for solving our problem and a higher resolution image of the crust and upper mantle velocity structure will be required for investigating the deformation and movement of the upper mantle by numerical modeling.

From April 2003 to September 2004, State Key Laboratory of Earthquake Dynamics, Institute of Geology, China Earthquake Administration, deployed a passive seismic array consisting of 60 broadband stations (Fig. 1a). It can be seen from Fig. 1a that this array is composed of three profiles. One of them is the profile of ~500 km long from Kuche to Kuitun across the Chinese Tian Shan. Li et al. (2007) investigated the S-wave velocity structure of the crust and upper mantle down to 100 km along this profile by using the non-linear receiver function inversion technique (Liu et al., 1996). Their results show that the crust beneath the Chinese Tian Shan has a laterally blocked structure and the averaged thickness of the crust is ~50 km.

From the travel time data of the P-arrivals recorded by this movable seismic array and permanent stations of the Xinjiang regional seismic network, Guo et al. (2006) presented the P-wave perturbation image of the crust and upper mantle down to the depth of 400 km along the Kuche–Kuitun profile by using the travel time tomographic technique (Zhao et al., 1992, 1994). The lateral resolution of their results reaches to 55 km. However, the velocity model of the upper mantle down to the depth of 660 km will be expected to simulate the deformation and movement of the upper mantle. For this purpose, it is reconstructed in this study using the same data and same technique. To reserve the original resolution, the velocity structure from the ground surface to the depth of 400 km is fixed in our reconstruction procedure.

Fig. 1b shows the P-wave velocity perturbation of the crust and upper mantle along the Kuche–Kuitun profile and the Moho boundary given by the receiver function study (Li et al., 2007). Fig. 1b also shows the background P-wave velocity, which is taken from the IASP91 model (Kennett and Engdahl, 1991). It should be noted that the P-wave velocity in the IASP91 model is increased with depth. It can be seen from Fig. 1b that underneath the Tarim and Junggar basin exist obvious low-velocity anomalies in the depth of 150–450 km and a clear high-velocity anomaly in the depth of 60–250 km, which thrusts with a dip angle of ~30° southward from the Junggar basin and extends intermittently to the depth of ~600 km. The maximal P-wave velocity perturbation reaches to 6% in the upper mantle along this profile, which implies that large density or temperature anomalies exist in the upper mantle beneath the Chinese Tian Shan. This encourages us to study the possibility of the small-scale convection driven by the density anomalies beneath the Chinese Tian Shan.

3. Method

Since 1980s, the finite element method (FEM) has become one of important tools of investigating the mantle convection (Christensen, 1984; King et al., 1990; Moresi and Gurnis, 1996; Zhong et al., 2000). Usually, the creeping flow problem is solved with the Eulerian description. Due to the mesh grids are fixed in the Eulerian description, the position of material at the current nodal point will be difficult to be determined in the next moment. Thus, the Eulerian description is hard to be used for determining the time-dependent shape variations in heterogeneous media.

The so-called marker-in-cell (MIC) or particle-in-cell (PIC) algorithm for the Eulerian method was proposed firstly by Harlow and Welch (1965). Their main idea is that the micro-elements called the markers or particles are filled in cells to mark the material property. In terms of the marker’s movement, the material deformation is described easily. These have been used in solving the geodynamic problems (Weinberg and Schmeling, 1992; Schott and Schmeling, 1998). The numeric technique used could be the FEM or the finite difference method (Gerya and Yuen, 2003; Moresi et al., 2003; Tackley and King, 2003).

A new technique of combining the FEM with the MIC algorithm was proposed by Liu et al. (2006), which has been verified with benchmarks (Travis et al., 1990). In terms of the modularized design, their computation program is easily extensible and will be used in this study. The main idea of this method is given in Appendix A.

4. Model and boundary conditions

In this section, we shall discuss the earth model used for our numerical simulation. In principle, the numerical simulation of the mantle convection should be a 3D problem. However, due to lack of high-resolution 3D seismic tomographic data, our work has to be limited to the 2D case.

Our P-wave velocity model of the crust and upper mantle along the Kuche–Kuitun profile is derived from
the P-wave velocity perturbation and the IASP91 model shown in Fig. 1b. The crustal thickness of our model is 50 km. Then, the density distribution in the crust and upper mantle can be estimated further from the P-wave velocity model in terms of the Birch’s law (Birch, 1960, 1961):

\[ \rho_c = 0.2676V_p + 1.1733, \]
\[ \rho_m = 0.3413V_p + 0.5588 \]

where \( V_p \) denotes P-wave velocity, \( \rho_c \) and \( \rho_m \) denote the density of the crust and upper mantle, respectively. The coefficients in the Eq. (1) can be obtained from the IASP91 model (Kennett and Engdahl, 1991). It should be noted that the density is increased with depth in our crust and upper mantle model.

However, as mentioned above, the length of our profile is only 550 km long. Our numerical tests demonstrate that such a size is not enough, so that the obvious boundary effects will appear in the upper mantle convective simulation. Thus, we have to enlarge the horizontal size of our density model to avoid this problem. For this purpose, the density contours on both sides are stretched horizontally.

Our density model used in this study is shown in Fig. 2, from which we can see that the size of our density model reaches to 990 km in the horizontal direction and the aspect ratio reaches to 1:1.5. The thick line in Fig. 2 indicates the bottom boundary of the lithosphere determined according to the density contour of 3.4 g/cm³. It should be pointed out that this is a result after smoothing in order to restrain its undulations. It can be seen from Fig. 2 that the lithospheric thickness in our model is \( \sim 140 \) km beneath the mountain range and it reaches \( \sim 240 \) and \( \sim 250 \) km beneath the Tarim and Junggar block, respectively. It is known that the lithospheric thickness at the continental root is about 150–250 km (Schubert et al., 2001). According to the observations of the S-wave receiver functions, Kumar et al. (2005) inferred that the lithospheric thickness is 90–120 km beneath the western Tian Shan and 160–270 km underneath the Tarim block. Thus, our result looks reasonable. In addition, Fig. 2 also shows obvious lateral variations of the density underneath the Tian Shan, which are consistent with the large P-wave velocity perturbations in the same region shown in Fig. 1b.

Usually, in the mantle convection studies, the density model is derived from the temperature and the equation of state (EOS). As mentioned above, however, our density model is derived from the P-wave velocity model given by seismic tomography. Thus, our initial temperature distribution can be derived from a normal temperature model in the crust and upper mantle (Jeanloz and Morris, 1986; Schubert et al., 2001; Stuwe, 2002). In particular, we assume that the temperature of 1200 °C at the lithospheric bottom and a temperature difference of 1700 °C between the ground surface and bottom of the upper mantle. Then, in terms of the linear interpolation, the temperature distribution in the crust and upper mantle can be estimated further. As shown in Fig. 3a, we can see the temperature distribution increased with depth in the upper mantle.

In general, it is uneasy to have an accurate estimation of the mantle viscosity. Usually, the mantle viscosity of \( 10^{19} \) to \( 10^{22} \) Pa s has been recognized and its lower limitation corresponds to the value for the asthenosphere (King, 1995). The viscosity of \( 10^{21} \) Pa s proposed firstly by Haskell (1935, 1936) also has been accepted widely and called the characteristic viscosity of the mantle. The averaged viscosity over the whole mantle is \( 3 \times 10^{21} \) Pa s (Karato, 2003). Recently, however, some of new results show that the viscosity of the upper mantle is \( 10^{19} \) to \( 10^{21} \) Pa s and its averaged value is \( 4 \times 10^{20} \) Pa s (Forte and Mitrovica, 2001; Mitrovica and Forte, 2004). Thus, the viscosity of \( 10^{19} \) to \( 10^{21} \) Pa s could be taken as an accepted estimation for the upper mantle and its upper limitation can reach to \( 3 \times 10^{21} \) Pa s. The results mentioned above can be taken as our constraints of the viscosity model in this study.

It is well known that the viscosity depends on several factors, including temperature, pressure and stress, etc. Among these, the temperature and pressure are more important than others (Turcotte and Schubert, 1982). Thus, in our case, we only consider the temperature and pressure. Following Christensen (1984), the dimensionless viscosity of the crust and upper mantle can be
estimated using the formula:

\[ \eta = \frac{1}{\eta_0} \exp \left[ \frac{E + W(1 - z)}{2.088 + T} \right] \]  

(2)

where \( \eta_0 \) is the reference viscosity; \( E \) and \( W \) denote the dimensionless activation-energy and activation-volume, respectively; \( T \) denotes the dimensionless temperature; \( z \) is the dimensionless scale in the vertical direction, which is zero at the bottom and equals 1 at the top of our model. It can be seen from Eq. (2) that the viscosity is decreased, when the temperature is increased, and increased with depth or pressure.

In this study, the parameters of \( E \) and \( W \) are taken from those given by Christensen (1984). The former studies show that the viscosity near the ground surface is \( \sim 10^{25} \) Pa s (Schott and Schmeling, 1998; Burov and Guillou-Frottier, 2005). Our computational results by Eq. (2) are close to this value.

The GPS observations demonstrate that the Chinese Tian Shan crust has a shortening rate of 7–8 mm/year along the Kuche–Kuitun profile (Wang et al., 2000). To simulate this shortening crust, a lateral variation of the crustal viscosity is assumed. For this purpose, the viscosity of the crust beneath the Tian Shan range is assumed to be only 10% of the result given by Eq. (2). This implies that we have a weakened crust beneath the Tian Shan range and solid basins on both sides. This assumption is mainly based on the GPS data and our numerical tests. Although this assumption is something subjective, it is still reasonable and feasible for our study. This is because that the crust does not involve the mantle convection and the simulation of the crustal deformation is not our main purpose in this study.

Fig. 3b shows our viscosity model of the crust and upper mantle. It should be pointed out that Fig. 3b is merely one of different possible viscosity distributions along the profile, which have been considered also relevant to different lithospheric thickness, temperature distributions as well as the parameters in Eq. (2). Our tests manifest that the result shown in Fig. 3b seems to be more reasonable than others.

On the top of our model, a free boundary is placed, which corresponds to the ground surface, and in the bottom and two side-boundaries are set the free-slip boundary. The temperatures at the top and bottom are fixed in our model. The thermal fluxes on two side-boundaries are assumed to be zero.

5. Results

Fig. 4a and b shows our computational results of the small-scale convection underneath the Chinese Tian Shan based on two different models, unextended and extended model, respectively. In these computations, the boundary conditions without the external force are applied to the model. It can be seen from Fig. 4 that obvious boundary effects appear in Fig. 4a and this phenomenon does not appear in Fig. 4b, from which we can see that the main part of the convection cell is limited merely to the unextended range of the model. Comparing Fig. 4a with Fig. 4b, we find that the convection cells for these two models are quite similar each to other in the unextended range.

In addition, Fig. 4b also illustrates that underneath the Junggar basin and north Tian Shan exists a southward- counterclockwise convection and its convective scale reaches to \( \sim 500 \) km in both of vertical and horizontal direction. In addition, a weak northward-clockwise convection exists underneath the Tarim basin and south Tian Shan. These two convection cells merged nearby the south border of the mountain range. This implies that due to the drag effects of the mantle convection,
Fig. 4. The upper mantle convection driven by density anomalies: (a) unextended model; (b) extended model. Arrows denote the vectors of convection velocity. The magnitude of velocity is scaled by the length and colour of the arrows. Black lines indicate the density contours in the upper mantle at the initial state and the bounds of mountains and basins in the crust.

Since our viscosity model is Newtonian, the speed of the mantle flow is inversely proportional with the viscosity (Fowler, 1985). When all of parameters of our model used are kept same, except the viscosity, our numerical tests demonstrate that the upper mantle convection can have a similar pattern for different viscosity models, although the convective speed can have different magnitude. In particular, when taking a homogeneous viscosity model with the viscosity of $3 \times 10^{21}$ Pa s for the upper mantle and assuming that the viscosity of the crust is $10^{24}$ Pa s in the mountain range and $10^{25}$ Pa s in the basin area, our computational results demonstrate that the convective speed at the top of the upper mantle beneath the Tian Shan can reach to $\sim 20$ mm/year. According to these results, we infer that the greatest convective speed in the upper mantle underneath the mountain range should not be less than 20 mm/year for any normal viscosity models.

Fig. 5a shows the horizontal velocity distribution on the ground surface relevant to Fig. 4b and manifests that an opposite movement exists between the Tarim and Junggar block, and the shortening rate of the crust in the mountain range is less than 1 mm/year. This is inconsistent with the GPS data (Wang et al., 2000). The corresponding vertical velocity distribution is shown in Fig. 5b, in which the mountain range has a descending movement consisting with the down-welling of the upper mantle. This is also contradictory with the geodesy and geological observations (Peng, 1993; Deng et al., 2000), which show an apparent contemporary uplift. This illustrates that the uplift of the Tian Shan does not rely merely on the upper mantle convection.

In practice, it has been recognized that the Tarim block plays an important role in the uplift of the Tian Shan.
Fig. 6. The upper mantle convection when considering the extrusion of the Tarim (annotations are the same as Fig. 4).

(Avouac et al., 1993; Deng et al., 2000). Therefore, it will be necessary to consider the northward movement of the Tarim in our computation.

According to the GPS data (Wang et al., 2000), we assume that the Tarim crust has a northward movement at the velocity of 7.17 mm/year, which is equal to the dimensionless velocity of 150 in our computation. The result under this boundary condition is shown in Fig. 6, from which we can see that the upper mantle convection is almost no difference from that in Fig. 4b, except that the movement of the Tarim lithosphere becomes slightly faster. This manifests that the northward movement of the Tarim crust nearly does not influence the upper mantle convection. A reason is that the Tarim block is far from the main cell of the upper mantle convection. Another reason is that the movement speed of the Tarim block is much smaller than the mantle flow. When the same boundary condition is extended to the whole lithosphere, our numerical tests demonstrate that the mantle convection is nearly unchanged.

In addition, Fig. 6 also shows that the lower lithosphere of the Junggar block has a component of the southward movement dragged by the convection of the underneath mantle. This could resist the northward movement of the Tarim block. Simultaneously, the lower part of the Tarim lithosphere is impeded and dragged downward by the down-welling of the underneath mantle.

Fig. 7 shows the velocity distribution on the ground surface relevant to Fig. 6. In Fig. 7a is given the comparison of the computational horizontal velocity distribution with the GPS data at 18 stations (Niu et al., 2005), which have been projected on the Kuche–Kuitun profile. Fig. 7a manifests that the simulated deformation of the crust has a tendency similar with the GPS data. This demonstrates our boundary condition is reasonable.

Fig. 7b shows the velocity of the simulated vertical movement on the surface, which is completely different from the case shown in Fig. 5b. It can be seen from Fig. 7b that the uplifting rate is increased step by step from the south Tian Shan to north Tian Shan and the maximum rate reaches to \( \sim 1.13 \) mm/year. This result is consistent with the leveling survey (Peng, 1993). Comparing Fig. 7b with Fig. 5b, we can see that the northward movement of the Tarim block plays the key role in the Cenozoic uplift of the Tian Shan Mountains.

According to the computed principal strain-rate in the mountain range over the depth of 0–120 km (Fig. 8), we found that both of the vertical and horizontal crustal deformation in the north Tian Shan are larger than those in the south Tian Shan. This could be caused by the thin lithosphere underneath the mountain range and the large convective velocity of the underneath upper mantle. Meanwhile, the results shown in Fig. 8 can be used for explaining why the north Tian Shan has a greater uplifting rate than the south Tian Shan (Peng, 1993). This implies that the small-scale upper-mantle convection beneath the Tian Shan not only resists the northward movement of the Tarim block, but also distorts the crust through the heated and thinned lithosphere.

It should be emphasized that all of results shown in Figs. 4–8 just represent the output at the first-time step for a transient problem, which corresponds to the result obtained from the contemporary geophysical data.
In this study, we also calculated the evolution of the upper mantle convection, although it is less important. For the convection shown in Fig. 4b, the average kinetic energy will decrease exponentially and remain \( \sim 1\% \) after 1.2 Myears, and the maximal velocity in the profile will be decreased from 190 to 40 mm/year. Simultaneously, the density contour lines will be flattened significantly and the variations of the temperature distribution are similar with those of the density contours. In fact, due to that the northward movement of the Tarim block does not influence the upper mantle convection, the evolution of the convection shown in Fig. 6 is nearly same with that in Fig. 4b.

6. Conclusion and discussion

Based on the tomographic results given by Guo et al. (2006), we reconstruct the seismic tomographic images of the P-wave velocity structure of the crust and upper mantle along the Kuche–Kuitun profile. This new result covers the whole upper mantle beneath the Chinese Tian Shan and makes it possible to build a high-resolution model of the crust and upper mantle, which is used further to simulate the deformation and flow of the upper mantle beneath the Chinese Tian Shan.

In this purpose, the FEM combining with the MIC technique is used. Our results show that the small-scale convection of the upper mantle driven by density anomalies should exist underneath the Chinese Tian Shan. Based on our results, we can have the following conclusions:

(1) The small-scale convections underneath the Chinese Tian Shan are composed of two convective cells: one is the weak northward-clockwise convection underneath the Tarim and south Tian Shan; the other is the strong southward-counterclockwise convection underneath the Junggar and north Tian Shan, which scale reaches to \( \sim 500 \) km.

(2) The northward movement of the Tarim block nearly does not influence the upper mantle convection beneath the Chinese Tian Shan and the convective speed at the top of the upper mantle underneath the mountain range should not be less than 20 mm/year for a normal viscosity model of the upper mantle.

(3) The northward extrusion of the Tarim block plays a key role in the Cenozoic orogeny of the Chinese Tian Shan, and the present-day tectonic deformation of the orogeny is related to the upper mantle convection.

It should be point out that our crustal model is relatively simple in our computations. Moreover, our computations are limited to the 2D case. Especially, our discussions deal only with the crust and upper mantle along the Kuche–Kuitun profile, which is almost perpendicular to the mountain strike. In general, in the young orogenic belt, the deformation of the upper mantle should be parallel with the mountain strike (Meissner et al., 2002). Chen et al. (2005) also reported the results about the S-wave splitting from the data recorded by a movable array across the Chinese Tian Shan located at 86°E–88°E, suggesting a fast direction roughly parallel with the mountain strike along their observational profile. Therefore, the 3D small-scale convection beneath the Tian Shan remains still an open question. For this purpose, the 3D velocity structure of the crust and upper mantle will be required.
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Appendix A

A.1. Governing equations and solving strategy

The physical process of the upper mantle convection can be described by the conservation equations of mass, momentum and energy (Schubert et al., 2001). We assume that the medium is incompressible and the Prandtl number is infinite. In this study, we adopt the Boussinesq approximation. Thus, the governing equations can be written as

\[
\frac{\partial \mathbf{v}_i}{\partial x_i} = 0, \quad (A1)
\]

\[
\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i \quad (A2)
\]

\[
\frac{\partial T}{\partial t} + \mathbf{v}_i \frac{\partial T}{\partial x_i} - \kappa \frac{\partial^2 T}{\partial x_i^2} = \frac{H}{\rho c} \quad (A3)
\]

where \( \mathbf{v}_i \) and \( x_i \) (\( i=1, 2, 3 \)) are the velocity vector and coordinates, respectively; \( p \) is the hydrodynamic pressure; \( \sigma_{ij} \) is the deviatoric stress tensor; \( \rho \) is the density related to the composition and temperature; \( g_i \) is the \( i \)th component of gravitational acceleration; \( T \) and \( t \) represent the temperature and time, respectively; \( H \) is the radioactive heating production in an unit volume; \( \kappa \) is the coefficient of thermal diffusion, \( \kappa = k/\rho C \), \( k \) is the thermal conductivity, \( c \) is the specific heat. The deviatoric stress tensor can be related to the strain rate \( \dot{E}_{ij} \) in terms of the constitutive equation:

\[
\sigma_{ij} = 2\eta \dot{E}_{ij} = \eta \left( \frac{\partial \mathbf{v}_i}{\partial x_j} + \frac{\partial \mathbf{v}_j}{\partial x_i} \right) \quad (A4)
\]

where \( \eta \) is the viscosity.

The relationship between the density and temperature can be described by the equation of state (EOS):

\[
\rho = \rho_0 [1 - \alpha (T - T_0)] \quad (A5)
\]

where \( \rho_0 = \rho(T_0) \) refers to the reference density, and \( T_0 \) is the reference temperature, \( \alpha \) is the thermal expansion coefficient.

To preserve the numerical stability, we use the dimensionless variables. The dimensionless form of the conservation equations can be obtained from the scale coefficients:

\[
x'_i = \frac{x_i}{D}, \quad t' = t \frac{k}{D^2}, \quad \mathbf{v}_i = \frac{D}{k} v_i, \quad p' = p \frac{D^2}{\eta_0 \kappa}, \quad \eta' = \frac{\eta}{\eta_0}, \quad T' = \frac{T - T_0}{\Delta T}
\]

where ' represents the dimensionless variable, \( D \) is the extent of the vertical direction, \( \eta_0 \) is the reference viscosity, \( \Delta T = T_1 - T_0 \) is the temperature difference between the top and bottom. It should be pointed out that the EOS (A5), as an independent equation, is not included in the governing equations and the density variable is still retained in the momentum equation during the dimensionless process in our method. This makes it possible to solve our problem, when the density distribution is not derived from the temperature and the EOS.

The governing equations are solved by using the FEM, and the medium movement can be described using the MIC technique. After initializing the element and marking mesh, the following operations will be performed at each time step: (1) solving the momentum equation at the known density; (2) solving the energy equation from the velocity distribution; (3) calculating the velocity and temperature of the markers; (4) determining the new position of the markers according to the step of velocity and time; (5) calculating the new nodal parameters of each element from the position and temperature of markers. These five steps will be performed iteratively, until the terminal condition, i.e. a steady state or a specified time, is satisfied.

A.2. Finite element scheme

The Stokes equation (A2) is solved simultaneously with the continuity equation (A1) given by the pressure-stabilizing Petrov–Galerkin (PSPG) method (Hughes et al., 1986), where the interpolations of the velocity and pressure are equal-order and the numerical stability can be preserved. The key point in this method is to perform additional perturbation stabilization for the pressure-term. For this purpose, we use the weight function:

\[
\delta \mathbf{v}_i + \frac{\alpha_c (h_c)^2}{2\eta} \frac{\partial \delta p}{\partial x_i} \quad (A6)
\]
when establishing the finite element equations by using the weighting residual method. Here \( h^2 \) is the size of the element, \( \alpha^2 \) is an adjustable constant.

We use the streamline upwind Petrov–Galerkin (SUPG) algorithm (Brooks and Hughes, 1982) to solve the energy equation. The weight function in the SUPG algorithm is

\[
\delta T + \frac{k}{||v||} \cdot v_i \frac{\partial \delta T}{\partial x_i} \quad (A7)
\]

where \( k \) is a variable relying on the size of the element, the velocity and the local Peclet number.

Based on the weight functions, we can have the weak-form of the governing equations. The computational codes can be generated automatically with the finite element program generator (FEPG). The discrete energy equation in time domain is solved in terms of the Crank–Nicolson algorithm.

### A.3. Marker-in-cell algorithm

Firstly, the velocity and temperature of markers within the element are obtained by interpolating the results of the FEM and the bi-linear interpolation function is used. In particular, for the rectangular elements, the interpolation function is

\[
B_m = \left(1 - \frac{\Delta x_m}{dx}\right) \left(1 - \frac{\Delta z_m}{dz}\right) B_i + \frac{\Delta x_m}{dx} \frac{\Delta z_m}{dz} B_k + \left(1 - \frac{\Delta x_m}{dx}\right) \frac{\Delta z_m}{dz} B_l \quad (A8)
\]

where \( B_m \) is the parameter of the \( m \)th marker, \( B_P (P = i, j, k, l) \) is the parameter at four nodes of each element where the marker is located, \( dx \) and \( dz \) are the scale of the element in the horizontal and vertical direction, respectively, \( \Delta x_m \) and \( \Delta z_m \) represent the distance of \( m \)th marker to the left-bottom corner of the \( i \)th node in the \( x \) and \( z \) direction, respectively.

The new position of the markers can be determined according to the moving speed of the markers calculated by using the Runge–Kutta scheme of the forth-order in the space and first-order in time.

The material density of a marker depends on its current temperature. This can be calculated by the EOS. Simultaneously, the material distribution in different position will be varied with the marker shift. Thus, it is necessary to recalculate the corresponding parameters (density, viscosity coefficient, etc.) at the Eulerian nodal points. The parameters at the \( P \)th node are obtained from the weighted average over all markers in surrounding cells (Gerya and Yuen, 2003). This can be made in terms of the formula:

\[
B_P = \frac{\sum_m B_m w_m(P)}{\sum_m w_m(P)} \quad (A9)
\]

where \( w_m(P) \) is the statistical weight function of the \( m \)th marker at the \( P \)th node. For the rectangular elements:

\[
w_m(P) = \left(1 - \frac{\Delta x_mP}{dx}\right) \left(1 - \frac{\Delta z_mP}{dz}\right) \frac{1}{dxdz} \quad (A10)
\]

where \( \Delta x_mP \) and \( \Delta z_mP \) represent the distance of the \( m \)th marker to the node \( P \) in the \( x \) and \( z \) direction, respectively.

### References


